MEMORANDUM
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ON A NEW APPROACH TO THE NUMERICAL SOLUTION OF A CLASS OF PARTIAL DIFFERENTIAL INTEGRAL EQUATIONS OF TRANSPORT THEORY

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PREFACE

This Memorandum is part of RAND's continuing search for new ways of utilizing the modern digital computer. The authors present a method for numerically integrating nonlinear partial differential integral equations, which occur in such fields as radiative transfer and mathematical biology. The method is then specifically applied to solving a basic equation of transport in a spherical shell.

SUMMARY

In this Memorandum, the authors show how to approximate a nonlinear partial differential integral equation by a system of ordinary differential equations. A table of necessary constants is provided, and the results of a test calculation on an equation of radiative transfer in a spherical shell are described.

CONTENTS

I. INTRODUCTION

In applying invariant imbedding to the radiative transfer processes associated with plane parallel regions, we encounter a functional equation of the form

$$\frac{\partial S}{\partial z}(z,v,u) + \left(\frac{1}{v} + \frac{1}{u}\right)S = g(u,v,z) \tag{1}$$

$$+ \lambda \left[1 + \frac{1}{2}\int_0^1 S(z,v,u')\frac{du'}{u'}\right]\left[1 + \frac{1}{2}\int_0^1 S(z,v',u)\frac{dv'}{v'}\right] ,$$

where $S(o,v,u) = 0$ (see Refs. 1, 2, and 3). Here $z \geq 0$, $0 \leq u, v \leq 1$. This can be approximated by means of a finite dimensional set of ordinary differential equations by introducing quadrature techniques. Write

$$\int_0^1 S(z,v,u')\frac{du'}{u'} \cong \sum_{i=1}^{N} w_i S(z,v,x_i)/x_i \quad , \tag{2}$$

$$\int_0^1 S(z,v',u)\frac{dv'}{v'} \cong \sum_{i=1}^{N} w_i S(z,x_i,u)/x_i \quad ,$$

where $x_1, x_2, \ldots, x_N$ are the N roots of the shifted Legendre polynominal, $P_N^*(x) = P_N(1-2x)$. Using these approximate relations and setting $S(z,x_i,x_j) = S_{ij}(x)$, Eq. (1) reduces to a finite dimensional system subject to initial conditions. This technique has be quite successful in practice, as evidenced by the results in the cited references.

If we turn to the study of corresponding transfer processes for spherical and cylindrical regions, we meet a much more formidable equation

$$\frac{\partial S}{\partial z}(z,v,u) + \frac{1-v^2}{zv}\frac{\partial S}{\partial v} + \frac{1-u^2}{zu}\frac{\partial S}{\partial u} + \left(\frac{1}{v}+\frac{1}{u}\right) S - \left(\frac{v^2+u^2}{v^2u^2}\right)\frac{S}{z} \tag{3}$$

$$= g(u,v,z) + \lambda \left[1 + \frac{1}{2}\int_0^1 S(z,v,u')\frac{du'}{u'}\right]\left[1 + \frac{1}{2}\int_0^1 S(z,v',u)\frac{dv'}{v'}\right].$$

In the following we shall briefly sketch an approximation technique which enables us to reduce the numerical solution of Eq. (3) to that of a finite system of ordinary differential equations, and shall also describe a sample calculation. More detailed results will be presented subsequently. The method has been applied successfully to a number of other classes of functional equations involving partial derivatives. Finally, we note that equations involving partial derivatives and integrals occur with great frequency in mathematical biology.[4-6]

An approach of Chandrasekhar's is described in Ref. 1.

II. DERIVATIVES AS LINEAR COMBINATIONS OF FUNCTIONAL VALUES

In order to generalize the approach to Eq. (1), we replace the partial derivatives S_u and S_v by linear combinations of the values of S at the points $u,v = x_i, x_j$, $i,j = 1,2,\ldots,N$. Given a function $f(x)$, we want an approximate relation of the form

$$f'(x_i) \simeq \sum_{j=1}^{N} a_{ij}\, f(x_j) \quad , \qquad i = 1,2,\ldots,N \tag{4}$$

To determine the coefficients a_{ij}, we ask, by analogy with the Gaussian quadrature formula, that Eq. (4) be <u>exact</u> if $f(x)$ is a polynominal of degree N-1 or less. To obtain a_{ij}, we use the test functions $f_m(x) = P_N^*(x)/\left[(x-x_m)\, P_N^{*\prime}(x_m)\right]$. A simple calculation then yields

$$a_{im} = \frac{P_N^{*\prime}(x_i)}{(x_i-x_m)\, P_N^{*\prime}(x_m)} \quad , \qquad i \neq m \tag{5}$$

$$a_{mm} = \frac{P_N^{*\prime\prime}(x_m)}{2P_N^{*\prime}(x_m)} = \frac{(1-2x_m)}{2(x_m^2-x_m)}$$

for $m = 1,2,\ldots,N$. In view of the symmetry of $P_N^*(x)$ about $x = .5$, it is clear that $a_{ij} = -a_{N+1-i,\, N+1-j}$, a result which yields both a useful check on the calculation of these parameters and a reduction in the size of the tables. A table of values of a_{ij} for N = 5,7,9 follows. These values were calculated by H. Kagiwada and verified by J. Jolissant.

Table 1

THE COEFFICIENTS a_{ij} FOR N = 5

```
     i = 1
-0.10134081E 02  0.15403904E 02 -0.80870874E 01  0.39207982E 01
-0.11035337E 01
     i = 2
-0.19205120E 01 -0.15167064E 01  0.48055013E 01 -0.18571160E 01
 0.48883323E-00
     i = 3
 0.60233632E 00 -0.28707765E 01 -0.35527137E-14  0.28707765E 01
```

Table 2

THE COEFFICIENTS a_{ij} FOR N = 7

```
     i = 1
-0.19136364E 02  0.30166068E 02 -0.18345136E 02  0.12020668E 02
-0.73554054E 01  0.37037909E 01 -0.10536210E 01
     i = 2
-0.30774001E 01 -0.32947313E 01  0.94826608E 01 -0.49141384E 01
 0.27743267E 01 -0.13485609E 01  0.37784329E-00
     i = 3
 0.73878691E 00 -0.37433740E 01 -0.97174703E 00  0.56413488E 01
-0.24639939E 01  0.10951929E C1 -0.29621352E-00
     i = 4
-0.36940283E-00  0.14803137E 01 -0.43048331E 01 -0.99475983E-13
 0.43048331E 01 -0.14803137E 01  0.36940283E-00
```

Table 3

THE COEFFICIENTS a_{ij} FOR N = 9

i = 1			
-0.30899183E 02	0.49462602E 02	-0.31847722E 02	0.23009713E 02
-0.16634325E 02	0.11463908E 02	-0.71444762E 01	0.36223711E 01
-0.10328869E 01			
i = 2			
-0.46321847E 01	-0.55540647E 01	0.15529632E 02	-0.88594615E 01
0.58950087E 01	-0.39077266E 01	0.23856884E 01	-0.11961277E 01
0.33923594E-00			
i = 3			
0.99779608E 00	-0.51953604E 01	-0.19666417E 01	0.90706996E 01
-0.46474057E 01	0.27969636E 01	-0.16303335E 01	0.79812006E 00
-0.22383800E-00			
i = 4			
-0.41927865E-00	0.17238123E 01	-0.52755643E 01	-0.72470224E 00
0.67044574E 01	-0.30840075E 01	0.16267280E 01	-0.76033820E 00
0.20889316E-00			
i = 5			
0.25654308E-00	-0.97060200E 00	0.22877170E 01	-0.56744949E 01
0.56843419E-11	0.56744949E 01	-0.22877170E 01	0.97080200E 00
-0.25654308E-00			

III. SAMPLE CALCULATION

Presented below are the results of a particular calculation of interest in determining the flux reflected by a spherical shell atmosphere. We approximated Eq. (3), with $g(u,v,z) \equiv 0$, by the system of ordinary differential equations

$$\frac{dS_{ij}(z)}{dz} + \frac{1-v_i^2}{v_i z}\sum_{k=1}^{N} a_{ik} S_{kj} + \frac{1-v_j^2}{v_j z}\sum_{k=1}^{N} a_{kj} S_{ik}$$

$$+ \left(\frac{1}{v_i} + \frac{1}{v_j}\right) S_{ij} - \frac{v_i^2 + v_j^2}{v_i^2 v_j^2}\,\frac{S_{ij}}{z} \tag{6}$$

$$= \lambda\left[1 + \frac{1}{2}\sum_{k=1}^{N}\frac{S_{ik}\,w_k}{v_k}\right]\left[1 + \frac{1}{2}\sum_{k=1}^{N}\frac{S_{kj}\,w_k}{v_k}\right],$$

$$z \geq a\ ,$$

with initial conditions $S_{ij}(a) = 0$.

We set $\lambda = 1$, usually the severest test, and integrated over z from a to $a + 3$ with the values $a = 100$, 500 and 1,000. The constant a is the inner radius of the spherical shell. For comparison with the plane parallel case in Ref. 1, we also printed out values of $r_{ij}(z) = S_{ij}(z)/4x_i$.

Some results are shown graphically in Fig. 1. Note that as the radius of the inner surface of the shell increases, the reflection function of the shell approaches that of the slab. This is one test

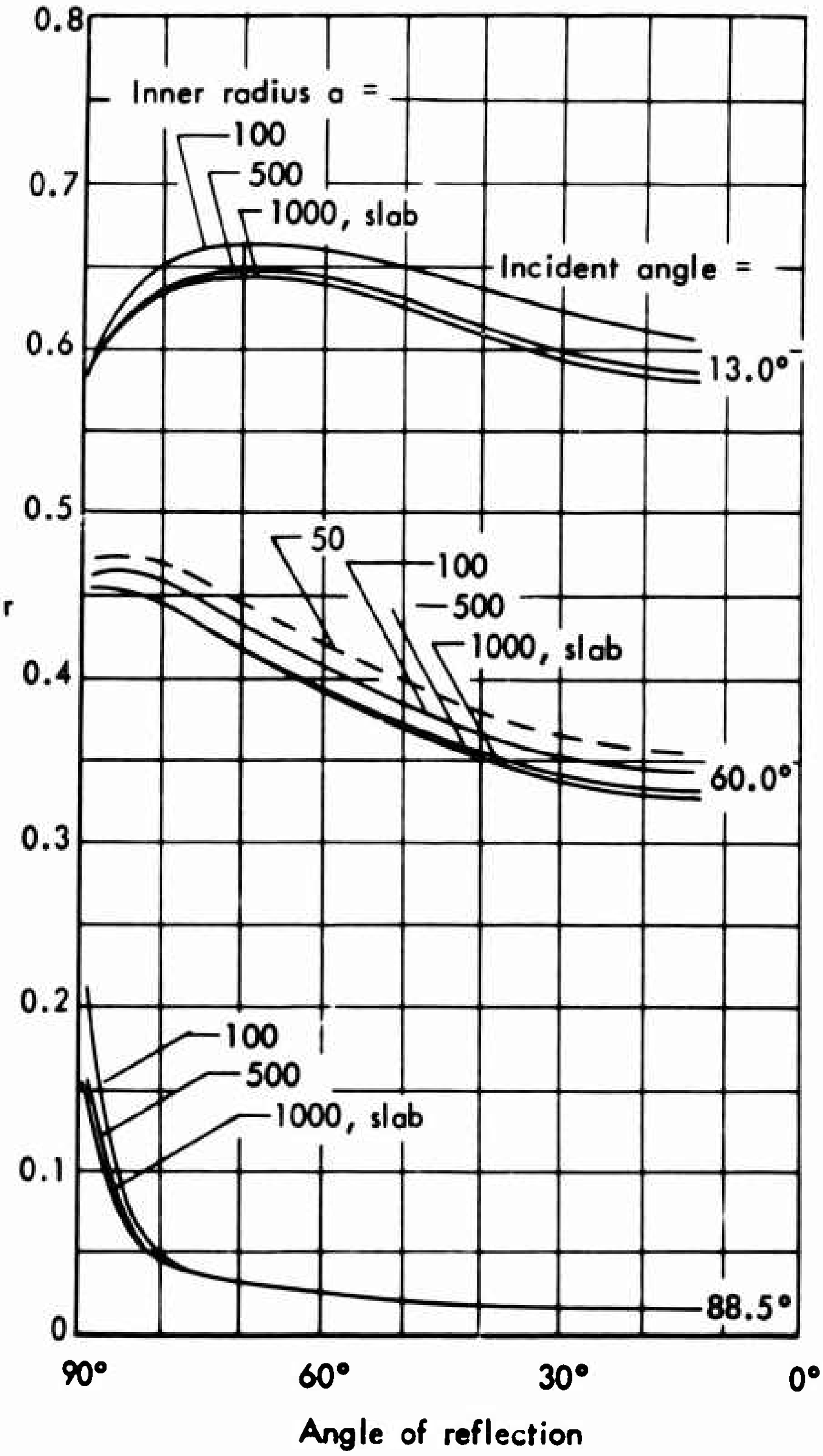

Fig. 1 — Some reflected intensity patterns for shells with albedo $\lambda = 1$ and thickness $x = 3$, for various angles of incidence

of the validity of the approximation technique used. In addition, comparison of the N=7 and N=9 cases indicates excellent agreement.

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